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The f-algebra Structure of Bi-orthomorphisms

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Özet

$A \times A$ üzerinde tanımlı $Orth(A,A)$ bi-ortomorfizmalar uzayının f-cebiri yapısını ve bir A vektör latisi üzerinde tanımlı $Orth(A)$ ortomorfizmalar uzayının $Orth(A,A)$ uzayının içine yerleştirilmesini araştırmaktayız. Özellikle, $Orth(A)$ ortomorfizmalar uzayında olduğu gibi, eğer A düzgün tam vektör latisi ise $Orth(A,A)$ uzayının bir düzgün tam vektör latisidir. Ayrıca, eğer A yarı-asal f-cebiri ise $Orth(A)$ uzayı $Orth(A,A)$'nın bir vektör altlatisi ve A yarı-asal Dedekind tam f-cebiri olduğunda, $Orth(A)$ uzayı $Orth(A,A)$'da bir ideal olduğunu elde edilir.

Anahtar Kelimeler: *f-cebiri, Ortomorfizma, Bi-Ortomorfizma*

Summary

We investigate the f-algebra structure of the space $Orth(A,A)$ of bi-orthomorphisms on $A \times A$ and the embedding of the space $Orth(A)$ of orthomorphisms on a vector lattice A into $Orth(A,A)$. In particular, as in the case of the space $Orth(A)$of orthomorphisms, we prove that if A is an uniformly complete vector lattice, then so is $Orth(A,A))$. Moreover, it is obtained that if A is a semi-prime f-algebra, then $Orth(A)$ is a vector sublattice of $Orth(A,A)$ and $Orth(A)$ is an order ideal in $Orth(A,A)$ whenever A is a semi-prime Dedekind complete f-algebra.

Keywords: *f-algebra, Orthomorphisms, Bi-orthomorphisms*

INTRODUCTION AND PRELEMINARIES

Definition 1.1. A (real) *vector lattice* (briey, an ℓ*-space*) or a *Riesz space* is an ordered vector space $\mathbf{A}$ with the additional property that for each pair of vectors $\mathbf{x,y \in A}$ the supremum, denoted by $x \vee y = sup\{x,y\}$, exists in A (or, equivalently, the infimum of the set $\{x,y\}$, denoted by $x \wedge y = inf\{x,y\}$ exists in A).

Definition 1.2. A vector lattice is called *Dedekind complete* whenever every non-empty subset which is bounded above has a supremum(or, equivalently, whenever every non-empty subset which is bounded below has an infimum).

Definition 1.3. An *order ideal* I in a vector lattice A is a vector subspace such that $y \in I, x \in A$ and $|x| \leq |y|$ together imply that $x \in I$.

Definition 1.4. A *band* B in a vector lattice A is an order ideal if any subset of B has a supremum in A and this supremum is an element of B; that is, if $D \subseteq B$ and $x = supD$, then $x \in B$.

Definition 1.5. A sequence $\{x_n\}$ in a vector lattice A is said to be *relatively uniformly convergent* to $x \in A$ if there exists some $0 < u \in A$ such that for every $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_n - x| \leq \varepsilon u$ for all $n \in \mathbb{N}$, $n \geq N_\varepsilon$.

Definition 1.6. A sequence $\{x_n\}$ in a vector lattice A is said to be *uniformly Cauchy* if there exists some $0 < u \in A$ such that for every $\varepsilon > 0$ there exists a natural number $N_\varepsilon \in \mathbb{N}$ such that$|x_n - x_m| \leq \varepsilon u$ for each $n,m \in \mathbb{N}$, $n,m \geq N_\varepsilon$.

Definition 1.7. A vector lattice is called *uniformly complete* whenever every uniformly Cauchy sequence is relatively uniformly convergent.

Definition 1.8. A vector lattice A is called *Archimedean* if given $0 \leq x,y \in A$ such that $nx \leq y$ for all $n \in \mathbb{N}$, it follows that $x = 0$.

Note that Dedekind completeness ⇒ uniform completeness ⇒ Archimedeaness.

All vector lattices under consideration are supposed to be Archimedean. For the elementary theory of vector lattices and terminology not explained here we refer to [1, 14].

Definition 1.9. A (real) vector lattice A is said to be a *lattice ordered algebra* (briey, an ℓ*-algebra*) or *Riesz algebra* if it is a linear algebra (associative, but not necessarily commutative or unital) such that $\mathbf{a,b\in A^{+}\Rightarrow ab\in A^{+}}$, where $\mathbf{A^{+}=\{x\in A:x\geq 0\}}$ is the positive cone of $\mathbf{A}$.

Definition 1.10. A lattice ordered algebra A is said to be an f*-algebra* (*function algebra*) if $a\wedge b=0\Rightarrow ac\wedge b=ca\wedge b=0$ for all $\mathbf{c\in A^{+}}$ (first appeared in a paper by Birkhoff and Pierce [4] in 1956).

Definition 1.11. A lattice ordered algebra A is said to be an *almost* f*-algebra* if $a\wedge b=0\Rightarrow ab=0$ (by Birkhoff [3] in 1967).

Definition 1.12. A lattice ordered algebra A is said to be a *pseudo almost* f*-algebra* if $a\wedge b=0\Rightarrow ab\wedge ba=0=0$ (by the second author [11] in 2014).

Definition 1.13. A lattice ordered algebra A is said to be an *pseudo* f*-algebra* if $a\wedge b$ is a nilpotent element of A then $ab=0$ (by Boulabier and Hadded [5] in 2003).

Definition 1.14. A lattice ordered algebra A is said to be an *generalized almost* f*-algebra* if $a\wedge b=0$ in A, then ab is an annihilator of A (by Chill [7] in 2009).

Although these various classes of algebras are distinct, there are relations between them; for example, from the definitions it is obvious that every f-algebra is both an almost f-algebra and a d-algebra, but not conversely. As a consequence of results given in [5] and [7], the following implications hold.

f*-algebra* $\Rightarrow$ *pseudo* f*-algebra* $\Rightarrow$ *almost* f*-algebra* $\Rightarrow$ *generalized almost* f*-algebra.*

Moreover, every almost f-algebra (and so pseudo f-algebra) is a pseudo almost f-algebra. Every d-algebra is a pseudo almost f-algebra. An almost f-algebra and a d-algebra are in general independent. However, if some conditions are imposed, then we have the following relationships between the two classes. Any d-algebra which is commutative or has positive squares $(\forall a, a^{2}\geq 0)$ is an almost f-algebra. Moreover, under certain conditions, these classes of lattice ordered algebras

turn out to be the same; that is, in any semiprime (associative) lattice ordered algebra the classes of f-algebras, almost f-algebras and d-algebras are equivalent.

Theorem 1.15. Let A be a semi-prime lattice ordered algebra, which is associative. Then the following statements are equivalent.

(i) A is an f-algebra.

(ii) A is a d-algebra.

(iii) A is an almost f-algebra.

(iv) A is a generalized almost f-algebra.

In particular, every semi-prime generalized almost f-algebra is a pseudo almost f-algebra.

Since a lattice ordered algebra with unit element is semi-prime we in particular have the following result.

Theorem 1.16. In a (associative) lattice ordered algebra A with unit element $e > 0$, the following statements are equivalent.

(i) A is an f-algebra.

(ii) A is an almost f-algebra.

(iii) A is a d-algebra.

(iv) A is a generalized almost f-algebra.

(v) A is a pseudo almost f-algebra.

Definition 1.17. Let A and B be vector lattices. A linear operator $T: A \to B$ is said to be order bounded if the image under T of an order bounded set in A is again an order bounded set in B. The operator T is called positive if $T(A^+) \subset B^+$; i.e., $T(a) \in B^+$ for all $a \in A^+$.

It is well-known that the ordered vector space $L_b(A,B)$ of all order bounded linear mappings of a vector lattice A into a Dedekind complete vector lattice B is a Dedekind complete vector lattice (see, for example, [1, Theorem 1.13]). In this case $\forall S, T \in L_b(A,B)$ and $\forall x \in A^+$,

$$(S \vee T)(x) = \sup\{S(y) + T(z) : y,z \in A^+, y + z = x\}$$
$$(S \wedge T)(x) = \inf\{S(y) + T(z) : y,z \in A^+, y + z = x\}.$$

It follows from the above that

$$T^+(x) = sup\{T(y) : 0 \leq y \leq x\}$$
$$T^-(x) = sup\{-T(y) : 0 \leq y \leq x\}$$
$$|T(x)| = \sup\{|Ty| : |y| \leq x\}.$$

Note that if $A = B$, then $L_b(A,B)$ is denoted by $L_b(A)$.

THE SPACE OF ORTHOMORPHİSMS

Definition 2.1. Let A be vector lattice. An order bounded linear operator $T : A \to A$ is called an *orthomorphism* if, for all $a,b \in A$, $a \perp b \Rightarrow Ta \perp b$, where $a \perp b$ means that $|a| \wedge |b| = 0$.

The collection of all orthomorphisms on a vector lattice A forms a vector space and is denoted by $Orth(A)$,

i.e., $Orth(A) := \{T \in L_b(A) : a \perp b \Rightarrow Ta \perp b\}$.

Clearly, $Orth(A)$ is a vector subspace of $L_b(A)$, and $Orth(A)$ with the ordering inherited from $L_b(A)$ is an ordered vector space in its own right.

The ordered vector space Orth(A) has the following remarkable property: It is a vector lattice under the pointwise algebraic and lattice operations (see [1]).

Theorem 2.2. If A is an Archimedean vector lattice, then $Orth(A)$ is an Archimedean vector lattice. In particular, the operations in $Orth(A)$ are given pointwise; that is, $\forall S,T \in Orth(A)$,

$$(S \vee T)(a) = S(a) \vee T(a)$$
$$(S \wedge T)(a) = S(a) \wedge T(a)$$

holds for all $0 \leq a \in A$.

Moreover, Orth(A) has a structure of f-algebra [1].

Theorem 2.3. If A is an Archimedean vector lattice, then $Orth(A)$ is an Archimedean f-algebra under multiplication by composition, possessing the identity operator I on A as a multiplicative identity.

THE ALGEBRAIC STRUCTURE OF BI–ORTHOMORPHISMS

Definition 3.1. Let A be an Archimedean vector lattice. An order bounded bilinear map $T : A\times A \to A$ is called a *bi-orthomorphism* if its partial maps are orthomorphisms; that is, the bilinear map $T : A\times A \to A$ is called a bi-orthomorphism if, given $a \in A$, two operators $T_a : A \to A$ and ${}_aT : A \to A$ defined in a natural way by putting $T_a(x) = T(x,a)$ and ${}_aT(x) = T(a,x)$ for all $x \in A$ we have $T_a, {}_aT \in Orth(A)$.

The set of all bi-orthomorphisms on A is denoted by $Orth(A,A)$. More precisely, a bilinear map $T : A\times A \to A$ is said to be a bi-orthomorphism if a separately order bounded bilinear map such that $x \wedge y = 0$ in $A \Rightarrow T(z,x) \wedge y = 0$ for all $z \in A^+$ (first appears a paper by G. Buskes, R. Page, Jr. and R. Yilmaz [9] in 2009). Bi-orthomorphisms on an Archimedean vector lattice A were first introduced and studied by R. Yilmaz and K. Rowlands in [13], where they were called quasi-orthomorphisms on A. This study was resumed by G. Buskes, R. Page, Jr. and R. Yilmaz in [9] where, among other results, some corrections were made. The following theorem is one of the main results in [9]. It describes the order structure of the set $Orth(A,A)$ of all bi-orthomorphisms on A.

Theorem 3.2. If A is an Archimedean vector lattice, then $Orth(A,A)$ is an Archimedean vector lattice. In particular, the lattice operations in $Orth(A,A)$ are given pointwise; i.e., $\forall S,T \in Orth(A,A)$ and $a,b \in A^+$

$$(T \vee S)(a,b) = T(a,b) \vee S(a,b)$$

$$(T \wedge S)(a,b) = T(a,b) \wedge S(a,b),$$

from which it in particular follows that

$$T^+(a,b) = T(a,b)^+$$

$$T^-(a,b) = T(a,b)^-$$

$|T|(a,b) = |T(a,b)|$.

It follows that $T \in Orth(A,A)$ is positive if and only if $T(a,b) \in A^+$ for all $a,b \in A^+$.

We collect some important properties of bi-orthomorphisms [6, 8].

Theorem 3.3. Let A be an Archimedean vector lattice.

(1) Every bi-orthomorphism T on A is orthosymmetric; that is, $a \wedge b = 0$ in $A \Rightarrow T(a,b) = 0$.

(2) Every positive bi-orthomorphism T on A is symmetric; that is, for all $a,b \in A$, $T(a,b) = T(b,a)$.

(3) Every positive bi-orthomorphism T on A is a lattice (Riesz) bimorphism; that is, for all $c \in A^+$, $a \wedge b = 0$ in $A \Rightarrow T(c,a) \wedge T(c,b) = 0$.

As a consequence of the above theorem we have

Corollary 3.4. Let A be an Archimedean vector lattice.

(1) If $T \in Orth(A;A)$ and $a \in A$, then $T(a^+,a^-) = 0$, and so $T(a,a) = T(|a|,|a|)$.

(2) If $T,S \in Orth(A,A)$ and $x,y,z \in A$, then

$T(S(x,y),z) = T(x,S(y,z)) = S(x,T(y,z))$.

We now investigate the f-algebra structure of the space $Orth(A,A)$ of all bi-orthomorphisms on a vector lattice A. The question when $Orth(A,A)$ was an f-algebra was first arisen in [9] and a partial solution was provided, imposing some extra completeness and approximation conditions, as follows.

Theorem 3.5. If A be an f-algebra which is also a Banach algebra with a minimal ultra-approximate identity (for definitions, see [13]), then $Orth(A;A)$ is a Banach f-algebra.

Recently, a paper published in 2016 [6] gives a complete answer to the above problem. It is proved that $Orth(A,A)$ can always be equipped with a non-trivial f-algebra structure, unless $Orth(A,A)$ itself is trivial.

Theorem 3.6. If A is a Archimedean f-algebra and $0 < e \in A$, then $Orth(A,A)$ is an

Archimedean f-algebra with respect to the multiplication $*_e$ defined by, for all $T,S \in Orth(A,A)$ and $x,y \in A$,

$(T *_e S)(x,y) = T(x,S(e,y))$.

Proof. Let A be a non-trivial f-algebra. For every $0 < e \in A$, the bilinear map $*_e : Orth(A,A)\ Orth(A,A) \times Orth(A,A)$ defined by, for all $T,S \in Orth(A,A)$ and $x,y \in A$,

$(T *_e S)(x,y) = T(x,S(e,y))$

gives an f-algebra multiplication on $Orth(A,A)$. In other words, the vector lattice $Orth(A,A)$ is an f-algebra with respect to the multiplication $*_e$. Indeed, observe first that the map $T *_e S$: A × A →A defined by for all $x,y \in A$

$T *_e S\ (x,y) = T(x,S(e,y))$

is bilinear. Also, if $x \in A$, then

$T *_e S(x,.) = T(x,.)oS(e,.) = T(x,S(e,.)) \in Orth(A)$

and

$T *_e S(x,.) = T(.,S(e,x)) \in Orth(A)$,

which give $T *_e S \in Orth(A,A)$.

Now if $T,S,Z \in Orth(A,A)$ and $x,y \in A$, then

$((T *_e S) *_e Z)(x,y) = (T *_e S)(x,Z(e,y))$

$= (T(x,S(Z(e,y)))$

$= T(x,(S *_e Z)(e,y))$

$= (T *_e (S *_e Z))(x,y)$

and so $(T *_e S) *_e Z = T *_e (S *_e Z)$.

On the other hand, the bilinear map $*_e : Orth(A,A) \times Orth(A,A) \to Orth(A,A)$ is positive. Therefore $(Orth(A,A); *_e)$ is a lattice ordered algebra. Moreover, it follows from the fact that every bi-orthomorphism is symmetric that the lattice ordered algebra $(Orth(A,A),*_e)$ is commutative. Therefore, $(Orth(A,A),*_e)$ is a commutative lattice ordered algebra.

Finally, in order to show that the lattice ordered algebra $Orth(A,A)$ is an f-algebra, let $T,S \in Orth(A,A)$ such that $T \wedge S = 0$. Then $\forall x,y \in A^+$

$T(x,y) \wedge S(x,y) = 0$.

Hence, $\forall\, 0 < Z \in Orth(A,A)$, we have

$Z(e,T(x,y)) \wedge S(x,y) = 0$,

as $Z(e,.) \in Orth(A)$. It follows from the symmetry of bi-orthomorphisms and the above equalities in Corollary 3.4 , we obtain

$Z(e,T(x,y)) = Z(x,T(e,y)) = (Z *_e T)(x,y)$.

Thus $Z *_e T(x,y) \wedge S(x,y) = 0$, and so $Z *_e T \wedge S = 0$. Also, $T *_e Z \wedge S = 0$ as $*_e$ is commutative. Therefore $*_e$ makes the lattice ordered algebra $Orth(A,A)$ an f-algebra; that is, $(Orth(A;A),*_e)$ is an f-algebra, as required.

As in the case of the space $Orth(A)$ of orthomorphisms, we can state more about the space $Orth(A,A)$ of bi-orthomorphisms. For details, we refer to [6].

Theorem 3.7. If A is an uniformly complete vector lattice, then so is $Orth(A,A)$.

Theorem 3.8. If A is a Archimedean vector lattice with a weak order unit $0 < e \in A$ $(a \in A;\ |a| \wedge e = 0 \Rightarrow a = 0)$, then the f-algebra $(Orth(A,A),*_e)$ is semi-prime.

For an Archimedean semi-prime f-algebra A, we can say more about the structure of the space $Orth(A,A)$. The mapping $\varphi : Orth(A) \to Orth(A,A)$, defined by

$(\varphi(\pi))(x,y) = \pi(xy)(= x\pi(y))\ (\pi \in Orth(A),\ x,y \in A)$,

is an injective Riesz homomorphism but, in general, not surjective;

i.e., $Orth(A) \neq Orth(A,A)$.

Indeed, take $A = C[0,1]$ and define an f-algebra multiplication by

$(f * g)(x) = x \cdot (f \cdot g)(x)$ $(f,g \in A$ and $x \in [0,1])$.

Then the ordinary multiplication $(f,g) \to f \cdot g$ is in $Orth(A,A)$ but not in $Orth(A)$.

Hence we see that $Orth(A)$ can be embedded in $Orth(A,A)$. In fact, in [9] if A is Dedekind complete, then it is proved that $Orth(A)$ is an order ideal in $Orth(A,A)$.

Theorem 3.9. Let A be a semi-prime f-algebra.

(1) $Orth(A)$ is a vector sublattice of $Orth(A,A)$.

(2) If A is Dedekind complete, then $Orth(A)$ is an order ideal in $Orth(A,A)$.

The above results has recently generalized in a paper published in 2016 [6].

Theorem 3.10. If A is an uniformly complete semi-prime f-algebra with a weak order unit, then $Orth(A)$ is a ring ideal, and so an order ideal in $Orth(A,A)$.

Remark. In generel, when $Orth(A)$ is a band in $Orth(A,A)$ is unclear. However, in a paper published in 2014 [2], under certain conditions, the necessary and sufficient conditions for $Orth(A)$ to be a band in $Orth(A,A)$ are given.

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